Free Semigroup Presentations

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Abstract
Let M^+ and N^+ be two free semigroups. We define external direct product of two free semigroups as an ordered pair of words $(w, u) \in (M \oplus N)^+$ such that $w \in M^+$ and $u \in N^+$. We investigate the presentations of external direct product of free semigroups, state and prove under some conditions that the external direct product of two finitely generated free semigroups is finitely generated, also the external direct product of two finitely presented free semigroups is finitely presented.

Keywords: *Semigroup; Free Semigroup; Presentations.*

1: Introduction

In 1998, Robertson et. al., [10], discussed the direct products of semigroups and provided the necessary and sufficient conditions for which the direct products of semigroups is finitely generated and presented. It was showed in [7] that using graph theory the direct product of two finitely generated (finitely presented) semigroups is also finitely generated (finitely presented). In [16], graphs were used to represent direct product of two and three finitely generated (finitely presented) semigroups. Further investigation by [15] proved that given a finitely generated semigroup, M and its subsemigroup (say) S then, if M is finitely generated and the subsemigroup S of M has a finite boundary in M, then S is finitely generated. Also, it was showed that if M is finitely presented and S has a finite boundary in M, then S is finitely presented. The presentations of semigroups and subsemigroups have extensively been discussed by different researchers. For details see [3], [7], [9], [10] and [11]. More so, different investigations have been independently and collectively carried out on the direct products of semigroups, see also [4], [5], [6], [7] and [19]. In this paper, we investigate the external direct products of free semigroups, state and prove the conditions for direct products of free semigroup to be free. In section 2, we give basic definitions relevant to our paper and prove in section 3 that the external direct product of free semigroups is finitely generated. In section 4, we state some conditions for the external direct product of semigroups to be finitely presented.

2. Preliminaries

In this section, we introduce the notations for the rest of the paper and give basic definition of terms that will be helpful as we proceed. For the notations see [1], [4], [5] and [6]. Readers are also encouraged to seek for detailed concepts explanations in the cited references.

Definition 2.1: A semigroup $(S, *)$ consisting of a set S and an associative binary operation$*$.
Examples 2.1: The integers, Z form a semigroup under two different operations: Addition, $(Z, +)$ and Multiplication $(Z, .)$

Definition 2.2: An Alphabet is an abstract set of symbols.

Note: Alphabet is denoted by A.
Example 2.2: A = {a, b, c, … }. The elements of the alphabet are called letter.

Definition 2.3: A word over A (the alphabet) is a string of letters $(a_1\, a_2 \ldots a_m)$ where each of the a_i is a member of A.
Example 2.3: $\{a, ab, \ldots\}$. The word "a" is a one letter-word while the word "ab" is a two letter word.

Remark 1: We denote the length of a word w by n or $|w|$ while a word of length zero (0) is called an empty sequence or an empty word. Also, the length of a word $|w| = 0$ if and only if $w = \varepsilon$. However, the set of words (including empty words) over A is denoted by M^*, The set of all nonempty words with at least length one (that is, it contains at least a one-letter word) is denoted by M^+. We quickly observed that $M^+ \subset M^*$. The empty word is denoted by ε. So for a word (say) $\varepsilon, w \in M^+$; $w.\varepsilon = \varepsilon.w = w$. This means that, ε acts as an identity element. Also for arbitrary $w_1, w_2 \in M^+$, we note that $|w_1 w_2| = |w_1| + |w_2|$.
A binary operation is defined on M^+ by concatenation, that is,
Given $(a_1 a_2 \ldots a_m), (b_1 b_2 \ldots b_n) \in M^+$ then,

$$(a_1 a_2 \ldots a_m)(b_1 b_2 \ldots b_n) = (a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n) \in M^+ - - - - - - - - - - - (\dagger)$$

Definition 2.4: Let A be alphabet and M⁺, the set of all finite, nonempty words over A. Then M⁺ is a free semigroup with respect to the operation defined in equation (†) above.

Definition 2.5: A free monoid is defined as a free semigroup adjoins with an empty word, ε. That is, $M^* = M^+ \cup \{\varepsilon\}$.

Example 2.5 (a): Let (P, +) be a semigroup of natural numbers under addition. (P, +) is a free semigroup. Let $T = \{1\}$ be the generating set for (P, +). Consider the map $\alpha: T \to R$ (where R is any semigroup) which can be extended to $\bar{\alpha}: (\mathrm{N}, +). \to R$ defined by $\bar{\alpha}(n) = \alpha(1)^n$. Suppose we place a restriction on $(\mathrm{N}, +)$ then, $\bar{\alpha}$ is a homomorphism since for $n, m \in N$; $\bar{\alpha}(n + m) = \alpha(1)^{n+m} = \alpha(1)^n \alpha(1)^m = \bar{\alpha}(n).\bar{\alpha}(m)$

Example 2.5 (b): Let us consider (P, .) and we show that it is not free. Let $T \subseteq P$ and define $S = (P, +)$ and let $\alpha: T \to S$ defend by $\alpha(n) = n$; $\forall\, n \in T$. Consider the homomorphic extension $\bar{\alpha}: (P, .) \to (P, +)$ to be any homomorphism then $\bar{\alpha}(n) = \bar{\alpha}(1.n) = \bar{\alpha}(1) + \bar{\alpha}(n)$. But $\alpha(n) = n \Longrightarrow \bar{\alpha}(1) = 0$. Since the element "0" is not an element of $S = (P, +)$. Therefore, (P, .) is not free.

Definition 2.6: Let M^+ be a free semigroup and let A be its generating set. Then A is a generating set for M^+ if every word formed over A is made up of the products of the elements of A which are the letters.

Example 2.6 (a): Let $A = \{a, b\}$ be binary alphabet. Then $M^+ = \{a, b, ab, aab \ldots\}$. See that all the word formed over A is made up the elements of A. Thus, A is a generator.
Suppose S is a semigroup generated by A, then there exist a homomorphism $\alpha: M^+ \to S$, that is, for all $w_1, w_2 \in M^+, \alpha(w_1 w_2) = \alpha(w_1)\alpha(w_2)$

Definition 2.7: Let M^+be a free semigroup. Then $M^+$is said to be finitely generated if the generator is finite or if it has a finite generating set.

Example 2.7: Let $A = \{a, b\}$ binary alphabet and let $\{a, b, ab, aab \dots\}$ be set of words formed from the binary alphabet. Then set of words over A is finitely generated

Definition 2.8: Let S be a semigroup and let $A \subseteq S$. Then, $q = q_1 q_2 \dots q_n$is a factorization over A on a condition that each $q_n \in A$; $n = \{1, 2, \dots, n\}$. Suppose A is the generating set for S, then every element $q \in A$ has a factorization over A.
We note further that if every element of A has a unique factorization then S is a free semigroup. Otherwise, it is not a free semigroup.

Theorem 2.1 [5]: *Let S be a semigroup and $X \subset S$. Then S is freely generated if and only if every $x \in S$ has a unique factorization over X*■
Lemma 2.1: *Let M^+ and N^+ be two free semigroups. Then, the external direct products of M^+ and N^+, is a semigroup.*
Proof: we show first that it is a semigroup. It suffices to show that for arbitrary words $(w_1, w_2), (w_3, w_4), (w_5, w_6) \in (M \oplus N)^+$
$(w_1, w_2)[(w_3, w_4)(w_5, w_6)] = [(w_1, w_2)(w_3, w_4)](w_5, w_6)$
Since the binary operation defined on the free semigroup is by concatenation, we have that:

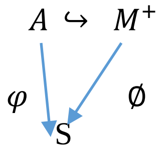
$$[(w_3, w_4)(w_5, w_6)] = (w_1, w_2)[(w_3 w_5, w_4 w_6]$$
$$= (w_1 w_3 w_5, w_2 w_4 w_6) \in (M \oplus N)^+ - - - - - - - - - - - - - - (1)$$

Similarly, $[(w_1, w_2)(w_3, w_4)] = [w_1 w_3, w_2 w_4](w_5, w_6)$

$$= (w_1 w_3 w_5, w_2 w_4 w_6) \in (M \oplus N)^+ - - - - - - - - - - - (2)$$

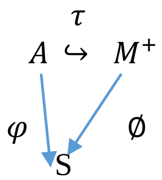
Hence, the two equations are equal, therefore $(M \oplus N)^+$is a semigroup.
We prove further that it is a free semigroup. Suppose $(w_a\, w_b \dots w_m), (v_a\, v_b \dots v_n) \in (M \oplus N)^+$ then we defined a concatenation on $(M \oplus N)^+$
We obtain $(w_a w_b \dots w_m)(v_a\, v_b \dots v_n) = (w_a w_b \dots w_m v_a, v_b \dots v_n)$
$(y_1 y_2 \dots y_m), (k_1 k_2 \dots k_n) \in (M \oplus N)^+$ then,
$(y_1 y_2 \dots y_m)(k_1 k_2 \dots k_n) = (y_1 y_2 \dots y_m k_1 k_2 \dots k_n)$
We see that $(w_a w_b \dots w_m v_a, v_b, \dots, v_n), (y_1 y_2 \dots y_m k_1 k_2 \dots k_n) \in (M \oplus N)^+$
Therefore the external direct product of two free semigroup is a free semigroup.
Example 2.8: Let $M^+ = \{a, ba, \dots\}$ and let $N^+ = \{c, cd, \dots\}$, then $(M \oplus N)^+ = \{ac, bacd, \dots\}$.

3 In this section, we state the necessary and sufficient conditions for the external direct products of two free semigroups to be finitely generated.
Definition 3.1: Let M^+ be free semigroup. We say that M^+ is decomposable if there is a word $w \in M^+$such that $w = w_1 w_2$ for some $w_1, w_2 \in M^+$.
Note: The set of decomposable free semigroup is denoted by $(M^+)^2 = M^+ M^+ = \{w_1 w_2 : w_1, w_2 \in M^+\}$
The set of indecomposable free semigroup is denoted by $M^+ \setminus (M^+)^2$
Example 3.1: Suppose $\{a, ab, aab, \dots\} \in M^+$is a set of words generated by the binary alphabet. The only indecomposable word is one-letter word in M^+.
Remark 3.1: Every indecomposable word belongs to a generating set. For instance, the one letter-word "**a**" is an indecomposable word in M^+but it belongs to the generating set—the binary alphabet. If A is a finite alphabet. By abstract definition of a free semigroup, M^+ is free on A if there is a map $\alpha: A \to M^+$
Lemma 3.1 [6]: *For every semigroup S and every map $\varphi: A \to S$ there exist a unique morphism $\emptyset: M^+ \to S$ such that diagram commutes*■



$$\tau\emptyset = \varphi$$

Lemma 3.2: *Let the diagram below be commutative. Then, M^+ is finitely generated.*



$$\tau\emptyset = \varphi$$

Proof: Suppose a finite alphabet A is given by $\{a_1, a_2, \dots, a_n\}$. Let $W = \{w_{1,} w_1 w_{3,} w_1 w_n \dots\}$ be words formed over A. Then, the homomorphism $\tau: A \longrightarrow M^+$defined by $\tau(a) = a_i b_j$ where $i = \{1,2, \dots, n\}$ and $j = \{1,2, \dots, n\}$ is an epimorphism since there exist a word (say) w such that $\tau(a) = a_i b_j = w$. Hence, since there is a morphism from A to M^+ we have established that M^+ is free on A. By lemma (3.1), the diagram commutes. This implies that every word formed over A is generated by the elements of A. and A is finite, thus $M^+$is finitely generated■

Lemma 3.3: *Let M^+ and N^+ be finitely generated free semigroups, then the external direct product $(M \oplus N)^+$ is finitely generated.*

Proof: Let A and B be finite alphabet such that A and B generates M^+ and N^+respectively. Denote the external direct product of A and B by $(A \oplus B) = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$ and let $W_1 = \{a_1 a_2 \dots a_n\}$ and $W_2 = \{b_1 b_2 \dots b_n\}$ be words formed over the alphabet A and B such that the map $\emptyset$ from $A \oplus B$ to $(M \oplus N)^+$ is an epimorphism. That is, there exist $(W_1, W_2) \in (M \oplus N)^+$such that $\emptyset(a_1, b_1) = (W_1, W_2)$ for all $(a_1, b_1) \in (A \oplus B)$. Thus, for every word say $(W_1, W_2) \in (M \oplus N)^+$ is being generated by $A \oplus B$. But $A \oplus B$ finite so $(M \oplus N)^+$ is finitely generated free semigroup follows from the fact that $\emptyset$ is an epimorphism■

Lemma 3.4: *Consider two finite free semigroups say M^+ and N^+. Given that $(M \oplus N)^+$is finitely generated then $(M^+)^2 = M^+ M^+$.*

Proof: Assume that $(M^+)^2 \neq M^+ M^+$. Then let $m \in M^+$ be an indecomposable element of M^+. So that for many pairs of word from $(w, n) \in (M \oplus N)^+$ will be indecomposable in $(M \oplus N)^+$.Since every indecomposable element belong to every generating set. This contradicts the assumption that $(M \oplus N)^+$is finitely.
Having stated and proved some important lemmas, we can now state and prove the main result of this section below:

Theorem 3.1: *Consider the two free semigroups M^+ and N^+. Then their external direct products $(M \oplus N)^+$is finitely generated if and only if M^+ and N^+ are finitely generated and with $(M^+)^2 = M^+ M^+$ and $(N^+)^2 = N^+ N^+$.*

Proof: Let A and B be the finite generating sets for M^+ *and* N^+respectively. Suppose $M^+$and $N^+$are finitely generated, then it follows from **lemma (3.3)** above that the extennal direct product $(M^+ \oplus N^+)$ is finitely generated. From **lemma (3.4)** it follows that $(M^+)^2 = M^+M^+$.Therefore both of the free semigroups is finitely generated. Thus, $(M^+)^2 = M^+M^+$ and $(N^+)^2 = N^+N^+$

Conversely, suppose M^+ *and* N^+ are finitely generated and $(M^+)^2 = M^+M^+$ and $(N^+)^2 = N^+N^+$. We show that $(M\oplus N)^+$is finitely generated. For $(M\oplus N)^+$to have a finite generator, it suffices to prove that there is an inclusion map $\emptyset: (A\oplus B) \hookrightarrow (M\oplus N)^+$ and a unique epimorphism $\varphi: (M^+\oplus N^+) \to S\oplus T$ where S and T are two semigroups such that $\emptyset\varphi = \theta$ where $\theta: A\oplus B \to S\oplus T$.

Let $(a_1, b_1), (a_1, b_1) \in A\oplus B$.

Then, $\emptyset(a_1,b_1)(a_2,b_2) = \emptyset(a_1a_2, b_1b_2) = (a_1a_2, b_1b_2) = \emptyset(a_1a_2), \emptyset(b_1b_2)$
$= \emptyset(a_1)\emptyset(a_2), \emptyset(b_1)\emptyset(b_2)$ ---------------------------(1)

since $\emptyset$ is a homomorphism.

Also, for arbitrary $(w_1, w_2), (w_3, w_4) \in (M\oplus N)^+$

Then, $\varphi(w_1, w_2)(w_3, w_4) = \varphi(w_1w_3), \varphi(w_2w_4) = \varphi(w_1)\varphi(w_3), \varphi(w_2)\,\varphi(w_4)$ ----(2)

Finally, for arbitrary $(a_1, b_1), (a_1, b_1) \in A\oplus B$.

$\theta(a_1, b_1)(a_2, b_2) = \theta(a_1a_2, b_1b_2) = (a_1a_2, b_1b_2) = \theta(a_1a_2), \theta(b_1b_2)$
$= \theta(a_1)\theta(a_2), \theta(b_1)\theta(b_2)$ -------------------------(3)

But $\emptyset\varphi$: $\theta: A\oplus B \to S\oplus T = \theta: A\oplus B \to S\oplus T$.

From $eqn(3)$, we observed that $\emptyset\varphi$ and θ preserved the same structure. Therefore, the diagram commutes and the prove is made ■

Definition 3.2: A free semigroup $M^+$is said to have a full generating set A if every word formed over A can be expressed as a product of two generators of A.

Example 3.2: Let $A = \{a, b\}$ then $M^+ = \{aa, ab, bb, \ldots\}$

Proposition 3.1 [9]: A semigroup M has a full generating set A if and only if M is decomposable. However, if M is finitely generated, then the generating set is finite ■

Corollary 3.1: *Let $M^+$and N^+be two decomposable free semigroups with full generating sets A and B respectively. Then $A\oplus B$ is a full generating set for $(M\oplus N)^+$Also if $(M\oplus N)^+$ is finitely generated, then the individual factors M^+ and N^+ are finite.*

Proof: Since A and B are full generating sets, it follows that every generator from A and B are product of two generators from A and B. Hence we can write that $A \subseteq A^2$ *and* $B \subseteq B^2$.

Algebraically, if $A \subseteq A^2$ and $B \subseteq B^2 \Longrightarrow A\oplus B \subseteq (A\oplus B)^2$

Hence, $A\oplus B$ is a full generating set. If $(M\oplus N)^+$is finitely generated, from theorem (3.1) it follows that the individual factors M^+ and N^+ are finitely generated. The result follows from the fact that A and B are finite ■

4 We have discussed generating sets for free semigroups in the previous section and their properties. One can't mention presentation without a generating set as presentations are defined in terms of generating sets and defining relations. In this section, we state and prove the necessary and sufficient condition for external direct products free semigroups to be finitely presented.

Definition 4.1: let A be a nonempty set (Alphabet). A free semigroup presentation is an ordered pair $\langle A|\mathfrak{R}\rangle$ where $\mathfrak{R} \subseteq M^+ \times M^+$ and A is a generating symbol.

Note: An element of $\mathfrak{R}$ (say) $(u, v) \in \mathfrak{R}$ is often written as (u=v)∈ $\mathfrak{R}$.

Remark 4.1: By a defining relation, we mean a relation in a free semigroup in which the elements commutes
To be precise, if $A = \{a_1, \dots, a_m\}$ and $\mathfrak{R} = \{u_1 = v_1, \dots, u_n = v_n\}$.
We write $\langle a_1, \dots, a_m | u_1 = v_1, \dots, u_n = v_n \rangle$ for $\langle A | \mathfrak{R} \rangle$.
Example 4.1: Suppose $A = \{a, b\}$ *and that* $M^+ = \{ab, ba, \dots\}$, then $\langle a, b\, | ab = ba \rangle$ defines the presentation for M^+.

Remark 4.2: Let w_1, w_2 in M^+ be arbitrary. Then $w_1 \equiv w_2$ if w_1 *and* w_2 are identical words in M^+. While $w_1 = w_2$ if both represent the same element in S. Let the two words w_1, w_2 in M^+ , w_2 is said to obtained from w_1 by one application of one relation if there exist $\alpha, \beta \in M^*$ such that for each $(u, v) \in \mathfrak{R}$; $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$. If $w_{1,}$ and w_2 are identical words we write $w_1 \equiv w_2$. For instance in the free semigroup defined by the presentation $\langle a, b | ab = ba \rangle$, we have that $aba = a^2 b$, but $aba \not\equiv a^2 b$. We say that w_2 can be deduced from w_1 if there exists a sequence $w_1 \equiv \alpha_1, \alpha_2 \dots \alpha_{k-1}, \alpha_k \equiv w_2$ of words from M^+ *such that* α_{i+1} is obtained from α_i by one application of one relation from $\mathfrak{R}$. Thus, $w_1 = w_2$ is a consequence of $\mathfrak{R}$. Let S be a semigroup with a generator B. Consider onto map φ from the alphabet B to the semigroup S and its unique extension *map* φ^1 from the free semigroup M^+ to a semigroup S. The semigroup S is said to satisfy relations $\mathfrak{R}$ on a condition that for each $u = v$ in $\mathfrak{R}$, we obtain that $\varphi(u) = \varphi(v)$. Any semigroup defined by a presentation $\langle A | \mathfrak{R} \rangle$ *is* M^+/ρ, where ρ the smallest congruence on M^+ containing $\mathfrak{R}$. By the smallest congruence on M^+, we mean the congruence for which M^+ is a free semigroup. That is, $\nexists \rho^1$ such that $\rho^1 \subset \rho$. Generally speaking, a semigroup S is said to be defined by a presentation $\langle A | R \rangle$ *if* $S \cong M^+/$p.

Proposition 4.1 [11]: *Let $\langle A | \mathfrak{R} \rangle$ be a presentation, let S be the semigroup defined by it, and let $w_1, w_2 \in M^+$. Then $w_1 = w_2$ in S if and only if w_2 can be deduced from w_1* ■

Proposition 4.2 [11]: *Let S be a semigroup generated by a set A, and let $\mathfrak{R} \subseteq M^+ \times M^+$. Then $\langle A | \mathfrak{R} \rangle$ is a presentation for S if and only if S satisfies all the relations from $\mathfrak{R}$. Secondly, if $u, v \in M^+$ are any two words such that S satisfies the relation u = v, then, u = v is a consequence of $\mathfrak{R}$* ■

Remark 4.3: Suppose $M^+ = \langle A_{M^+} | \mathfrak{R}_{M^+} \rangle$ *and* $N^+ = \langle B_{N^+} | \mathfrak{R}_{N^+} \rangle$ where A_{M^+} and B_{N^+} are the generating sets while $\mathfrak{R}_{M^+}$ and $\mathfrak{R}_{N^+}$ are the defining relations. Then $(M \oplus N)^+ = \langle A_{M^+} \cup B_{N^+} | \mathfrak{R}_{M^+} \cup \mathfrak{R}_{N^+} \cup \mathfrak{R}_t \rangle$ where $\mathfrak{R}_t$ is the set of relations that each element of A_{M^+} commutes with B_{N^+} .The presentation for $(M \oplus N)^+$is given by $\langle A \oplus B | \mathfrak{R} \rangle$ where A and B are the generating sets for M and N respectively. The structure of $(M \oplus N)^+$ can be used to construct the presentation for the external direct product of M^+ and N^+

Definition 4.2: Let M^+ be a semigroup define by the presentation $\langle A | \mathfrak{R} \rangle$. The semigroup M^+ is said to be finitely presented if it is defined by a finite presentation, that is, if A and $\mathfrak{R}$ are finite.
Example 4.2: The class of finitely presented free semigroups include: all finite free semigroups.

Definition 4.3: Let $\mathcal{B} = \langle A | \mathfrak{R} \rangle$ be a presentation and let M^+be a semigroup defined by it.
Consider two arbitrary words $w_{1,}, w_2$ *in* M^+. We say that the pair $(w_{1,} w_2)$ is called a critical pair for the free semigroup $M^+$with respect to the presentation $\mathcal{B}$ if the following conditions are satisfied:

1) w_1 and w_2 represent the same element in M^+. That is, $w_1 = w_2$
2) There exist $i(1 \geq i \geq k)$ such that $|\alpha_i| \leq \min(|w_1|,|w_2|)$ for every elementary sequence
   $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_k \equiv w_2$ with respect to the given presentation.

Definition 4.4: Consider a free semigroup with its generator M^+ and A respectively. Then, M^+ is said to be stable with respect to its generating set A if there exists a finite presentation $\langle A|\mathfrak{R}\rangle$ that defines M^+in terms of A with respect to which M^+ has no critical pairs.
We give below a very important property of a presentation in terms of a proposition.

Proposition 4.3 [10] *let $A = \{a_i : i\epsilon I\}$ and $B = \{ b_j : j\epsilon J \}$ be two finite generating sets for the free semigroup M^+. Suppose M^+ is stable with respect to its generating set A, it is as well stable with B as its generating set* ■

Theorem 4.1: *Let M^+ and N^+ be any two finite free semigroups. Then, $(M\oplus N)^+$is finitely presented if and only if neither M^+ nor N^+ has indecomposable elements and both semigroups have a finite presentation and are stable.*
Proof: Suppose that $(M\oplus N)^+$has a finite presentation, then we are certain that it also has a finite generation. Therefore, following the logic in theorem (3.3) above, we see that individual factors has a finite generation and has no indecomposable elements. The stability of the two free semigroups $M^+$and N^+ follows from the following proposition (4.4) while the converse of the theorem (4.1) follows from proposition (4.5).

Proposition 4.4: *Consider two decomposable free semigroups M^+ and N^+.Let $A = \{a_i : i\epsilon I\}$ and $B = \{b_j : j\epsilon J\}$ be full generating sets for M^+ and N^+respectively. Define $\mathfrak{B}$=$\langle A\oplus B|\mathfrak{R}\rangle$ a finite presentation for $(M\oplus N)^+$in terms of the generating set $A\oplus B$. Let $\pi_A : (A\oplus B)^+ \to A^+$ be a unique morphism extending the mapping $(a_i, b_j)^+ \mapsto (a_i)^+$ $(i\epsilon I, j\epsilon J)$. The presentation defined by S is given by $\mathfrak{D}$ =$\langle A|\pi_A(\mathfrak{R})\rangle$. Suppose further M^+ has no critical pairs with respect to the presentation $\mathfrak{D}$ =$\langle A|\pi_A(\mathfrak{R})\rangle$, then M^+ is stable if ($M^+ \oplus N^+$) is finitely presented.*

Proof: The fact that $M^+$and N^+ have full generating sets A and B respectively, follows from proposition (3.1) above that a semigroup M^+has a full generating set if and only if $(M^+)^2 = M^+M^+$. From corollary (3.1) we established that $A\oplus B$ generates $(M\oplus N)^+$
Suppose $(M\oplus N)^+$is finitely presented, the individual factors M^+ and N^+ are finitely generated follows from lemma (3.3) and A and B are finite.
The finite presentation $\langle A\oplus B|\mathfrak{R}\rangle$ defines the external direct product of the two free semigroups $(M^+ \oplus N^+)$, in terms of its generating set.
We prove further that $\mathfrak{D}$ =$\langle A|\pi_A(\mathfrak{R})\rangle$ is a presentation that defines M^+. It suffices to show that M^+ satisfies all the relations in $\mathfrak{R}$ and further we prove that every relation in M^+ is a consequence of $\mathfrak{D}$.
Let ϕ: $(A\oplus B)^+ \to (M\oplus N)^+$ denote a unique morphism that extends the map $A\oplus B \to (M\oplus N)^+$.Furthermore, let $\Phi_{m^+} : A^+ \to M^+$extending $A \to M^+$ and let π_{m^+}:($(M\oplus N)^+ \to M^+$be a natural projection
Since $\pi_A(\mathfrak{R})$ is a defining relation, and for $(u = v) \in \pi_A(\mathfrak{R})$ we see that $\pi_A(u)$ and $\pi_A(v)$ represent the same element in M^+
, that is, $\pi_A(u) = \pi_A(v)$ ------------------------------(a).

Applying Φ_{m^+} to the relation, in (a) above, we obtain:

$\Phi_{M^+}\,\pi_A$(u)= Φ_{M^+} ϕ(u)= Φ_{M^+} ϕ(v)= $\Phi_{M^+}\,\pi_A$(v) as required.
Next we show that for arbitrary two words $w_1, w_2 \in A^+$, Then, $w_1 = w_2$, if they represent the same element in M^+ and it is a consequence of $\mathfrak{D}$.
Consider two word w_1 and w_2 words and their equivalence written as;

$$w_1 \equiv a_{i_1} a_{i_2} \dots a_{i_m} \quad and \quad w_2 \equiv a_{k_1} a_{k_2} \dots a_{k_n}.$$

Then, there exist $b_{j_1} \dots b_{j_m}, b_{l_1} \dots b_{l_n}$ in $\mathfrak{B}$ such

$$(a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \dots (a_{i_m}, b_{j_m}) = (a_{k_1}, b_{l_1})(a_{k_2}, b_{l_2}) \dots (a_{k_n}, b_{l_n}) ------(1)$$

is a valid equation in $(M \oplus N)^+$ since (a_{k_n}, b_{l_n}) can generate $(M \oplus N)^+$
Assume $n \geq m, let\ b_{j_1} \dots b_{j_m}$ be arbitrary. Since $A \oplus B$ is a full generating set for $(M \oplus N)^+$ such that

$(a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \dots (a_{i_m}, b_{j_m})$ is equal to $(M \oplus N)^+$ with a word of length n

$$(a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \dots (a_{i_m}, b_{j_m}) = (a^1_{k_1}, b_{l_1})(a^1_{k_2}, b_{l_2}) \dots (a^1_{k_n}, b_{l_n}) ---------(2)$$

In the semigroup M^+ the following equation is true.

$$a^1_{k_1} a^1_{k_2} \dots a^1_{k_n} = a_{i_1} a_{i_2} \dots a_{i_m} \equiv w_1 = w_2 \equiv a_{k_1} a_{k_2} \dots a_{k_n}$$

It follows that in $(M \oplus N)^+$ we obtain,

$$(a^1_{k_1}, b_{l_1})(a^1_{k_2}, b_{l_2}) \dots (a^1_{k_n}, b_{l_n}) = (a_{k_1}, b_{l_1})(a_{k_2}, b_{l_2}) \dots (a_{k_n}, b_{l_n}) ------(3)$$

The combination of eqn(2) and eqn(3) yield the result below:

$(a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \dots (a_{i_m}, b_{j_m})$= $(a_{k_1}, b_{l_1})(a_{k_2}, b_{l_2}) \dots (a_{k_n}, b_{l_n})$ which is $eqn(1)$ and is valid in $(M \oplus N)^+$ and the claim is satisfied.
Let $w_3 = (a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \dots (a_{i_m}, b_{j_m})$ $and\ w_4 = (a_{k_1}, b_{l_1})(a_{k_2}, b_{l_2}) \dots (a_{k_n}, b_{l_n})$ represent the left and right side of $eqn(1)$.

Since $w_3 = w_4$ must be valid in $(M \oplus N)^+$ and $\mathfrak{B}$ a presentation for $(M \oplus N)^+$ there exist an elementary sequence

$$w_3 \equiv \alpha_1, \alpha_2, \dots \alpha_p \equiv w_4 ------------------------(4)$$

with respect to $\mathfrak{B}$.
Then,

$$w_1 \equiv \pi_A(w_3) \equiv \pi_A(\alpha_1), \pi_A(\alpha_2), \dots, \pi_A(\alpha_p) \equiv \pi_A(w_4) \equiv w_2 ----------(5)$$

is an elementary sequence with respect to D, such that for each $i\ (1 \leq i \leq k-1)$ either $\alpha_i = \alpha_{i+1}$ or α_{i+1} is obtained from α_i by one application of one relation $from$ $\pi_A(\mathfrak{R})$.
Then,
$\pi_A(\alpha_i) \equiv \pi_A(\alpha_{i+1})$. Where $\alpha_i \equiv \beta u \gamma$ and α_{i+1}= βvγ with $(u = v) \in \mathfrak{R}$ and $\beta, \gamma \in (M \oplus N)^*$
We certainly have,

$$\pi_A(\alpha_i) = \pi_A(\beta) \pi_A(u)\, \pi_A(\gamma),$$

$\pi_A(\alpha_{i+1}) \equiv \pi_A(\beta)\pi_A(v)\,\pi_A(\gamma)$, and $(\pi_A(u) = \pi_A(v)) \in \pi_A(\mathfrak{R})$,
Therefore, $w_1 = w_2$ is a consequence of $\mathfrak{D}$ as required.
Next, assume M^+ is infinite, and then we show that M^+ has not critical pairs with respect to $\mathfrak{D}$.
This can be achieved if we show that the elementary sequence in (5) contains no term shorter than min $(|w_1|, |w_2|)$. Since $b_{j_1} \dots b_{j_m}$ has been arbitrary chosen in the proof that $\mathfrak{D}$ is a presentation for M^+.
Let the word $w_5 \equiv b_{j_1} \dots b_{j_m}$ is not equal in N^+ to a shorter word and since N^+ is infinite, this can be done for any m.
Let $w_6 \equiv b_{l1} \dots b_{ln}$ and let $\pi_{B:}(A \oplus B)^+ \rightarrow B^+$ be the unique morphism extending the mapping
$(a_i, b_j) \mapsto b_j\ (i \epsilon I, j \epsilon J)$. By applying π_B to the elementary sequence in (4), the following relation is obtained:

$w_5 \equiv \pi_B(\alpha_1) = \pi_B(\alpha_2) = \ldots = \pi_B(\alpha_p) \equiv w_6$ which holds in N^+. Making a choice for a word w5 we have $|\pi_B(\alpha_1)| \geq |w_5| = m$ $(1 \leq l \leq p)$.
Conversely we obtain that $\pi_B(\alpha_1) = |\alpha_1| = \pi_A(\alpha_l) =$ $(1 \leq l \leq p)$ so that
$|\pi_A(\alpha_l)| \geq m = min(|w_1|, |w_2|)$.
Therefore, $(w_1, w_2) \in A^+$ is not a critical and their property (2) are satisfied.
The converse of theorem (4.1) follows from the following proposition.
Proposition 4.5 [9]: *Let M and N be decomposable and stable semigroups. Let $A = \{a_i : i \in I\}$ and $B = \{b_j : j \in J\}$ be finite and as well, full generating sets for M and Nrespectively. Define uniform presentation for M and Nrespectively with respect to which M and N have no critical pairs as:*

$$\mathcal{B} = \langle A | a_i = a_{\varphi(i)} a_{\phi(i)}, \Re \ (i \in I) \rangle$$
$$D = \langle B | b_j = b_{\varsigma(j)} b_{\Theta(j)}, Q \ (j \in J) \rangle$$

Suppose ξ denotes the decomposition mapping, then the external direct product $(M \times N)$ is defined by the following presentations.

$$\Re = \langle A \times B | \xi(u_1, \alpha) = \xi(v_1, \alpha) \rangle ----------------------(1)$$
$$\Re = \langle A \times B | \xi(\beta, u_2) = \xi(\beta, v_2) \rangle ----------------------(2)$$
$$\Re = \langle A \times B | \xi(a_{i_1} a_{\varphi(i_2)} a_{\phi(i_2)}, \gamma) = \xi(a_{\varphi(i_1)} a_{\phi(i_1)} a_{i_2}, \gamma) \rangle ---------- (3)$$
$$\Re = \left\langle A \times B \middle| \xi\left(\delta, b_{j_1} b_{\varsigma(j_2)} b_{\Theta(j_2)}\right) = \xi(\delta, b_{\varsigma(j_1)} b_{\Theta(j_1)}, b_{j_2}) \right\rangle -----------(4)$$
$$\Re = \langle A \times B | (a_i, b_j) = (a_{\varphi(i)}, b_{\varsigma(j)})(a_{\phi(i)}, b_{\Theta(j)}) \rangle ------------- (5)$$
$$\Re = \langle A \times B | ((u_1 = v_1) \in R;\ \alpha \in N^+;\ |u_1| = |\alpha|; (u_2 = v_2) \in Q;\ \beta \in M^+;\ |u_2| = |\beta|; \rangle$$
$$i_1, i_1 \in I;\ \gamma \in N^+;\ |\gamma| = 3;\ j_1, j_2 \in J; \delta \in M^+; |\delta| = 3; i \in I, j \in J.$$

Particularly,$M \times N$ is finitely presented■

Thus, we deduced the following facts.

- **(a)** M^+ is finitely generated if A is chosen to be finite
- **(b)** M^+ is finitely presented if the generating set A and the defining relation R can be chosen to be finite,
- **(c)** Every indecomposable word belongs to a generating set.
- **(d)** Not every finitely generated free semigroup is finitely presented e.g $\langle a, b | ab^i a = aba \ (i = 1, 2, 3, \ldots \rangle$
- **(e)** A finitely generated free semigroup $\nRightarrow$ finitely presented.
- **(f)** $(M^+)^2 \neq M^+ \Longrightarrow (M \oplus N)^+$is not finitely generated

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